

A NON-NORMAL FEFFERMAN-TYPE CONSTRUCTION OF SPLIT-SIGNATURE CONFORMAL STRUCTURES ADMITTING TWISTOR SPINORS

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ABSTRACT. We treat a non-normal Fefferman-type construction based on an inclusion $\mathrm{SL}(n+1) \hookrightarrow \mathrm{Spin}(n+1, n+1)$. The construction associates a split signature (n, n) -conformal spin structure to a projective structure of dimension n . For $n \geq 3$ the induced conformal Cartan connection is shown to be normal if and only if it is flat. The main technical work of this article consists in showing that in the non-flat case the normalised conformal Cartan connection still allows a parallel (pure) spin-tractor and thus a corresponding (pure) twistor spinor on the conformal space. The Fefferman-type construction presented here is an alternative approach to study a construction of Dunajski-Tod.

1. INTRODUCTION

The original Fefferman construction [Fef76] canonically associated a conformal structure on a circle bundle over a CR-structure. The resulting conformal structure is rather special: it admits solutions to certain invariant overdetermined equations, in particular, it carries a light-like conformal Killing field. In fact, it was shown by Sparling, cf. [Gra87], that a conformal structure is the Fefferman-space of some CR-structure if and only if it admits such a Killing field which also satisfies additional (conformally invariant) properties. This yields the characterisation of the CR-Fefferman spaces. The characterising property can alternatively be understood as a *holonomy reduction* of the conformal structure: It was shown in [ČG10] that a conformal structure (M, \mathcal{C}) is locally the Fefferman-space of a CR-structure if and only if its conformal holonomy satisfies $\mathrm{Hol}(\mathcal{C}) \subset \mathrm{SU}(p+1, q+1) \subset \mathrm{SO}(2p+2, 2q+2)$.

A generalisation of the original Fefferman-construction was described in [Čap05], and in recent years a number of constructions have been discussed in that framework: The original construction was treated via this approach in [ČG10], [HS09] discussed Nurowski's conformal structures [Nur05] that are associated to generic rank two distributions on 5-manifolds, [Alt10] treated a Fefferman-type construction of conformal structures from quaternionic contact structures, [HS11] discussed Bryant's [Bry06] conformal structures associated with generic 3-planes on 6-manifolds.

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In all cited cases the Fefferman-type construction is *normal*: this says that, starting from the normal Cartan connection encoding the original geometric structure (e.g., a CR-structure, a generic distribution, a quaternionic contact structure) the induced conformal Cartan connection form that is built via the Fefferman-type construction is again normal. This immediately implies that the holonomy of the conformal structure reduces to the included subgroup and makes it possible to derive a holonomy-based characterisation of the induced structures.

In this paper we discuss a non-normal Fefferman-type construction. We associate a split signature (n, n) conformal spin structure to a projective structure of dimension n . The construction is based on an inclusion $\mathrm{SL}(n+1) \hookrightarrow \mathrm{Spin}(n+1, n+1)$. If $n = 2$ this construction is shown to be normal, and the usual consequences on conformal holonomy reduction, Proposition 4.3, and symmetry-decomposition, 4.5, can be derived. In addition, it is also possible in this case to understand the space of (almost) Einstein metrics in the induced conformal class in terms of projective data, Proposition 4.4. For $n \geq 3$, the induced conformal Cartan connection is shown to be normal if and only if the original projective structure was already flat, Proposition 4.8. This fact immediately poses problems for the goal of relating the original projective and the induced conformal geometric structure: since the induced conformal Cartan connection form is not normal, its curvature and holonomy are no well defined conformally invariant objects. To obtain information on the conformal structure it is thus necessary to understand how the normal conformal connection differs from this one. We derive strong restrictions on the form of the normalised Cartan connection in Proposition 4.9. These imply in particular that the induced conformal structures, which carry a canonical spin structure, are endowed with a solution of the twistor spinor equation, Theorem 4.11.

The original motivation for this Fefferman-type construction comes from two sources. The first one is work by Dunajski-Tod, [DT10]: Extending a construction due to Walker [Wal54], which associates a pseudo-Riemannian split signature (n, n) -metric to an affine torsion-free connection on an n -manifold, they associate a conformal split signature (n, n) -metric to a projective class of torsion-free affine connections on an n -manifold. Using a normal form for the induced metrics it is also shown that they admit a twistor spinor. This construction is also discussed in Dunajski-West, [DW08]. The second source is a paper by P. Nurowski and G. Sparling, [NS03], which treats the construction from 2-dimensional projective structures to conformal structures of signature $(2, 2)$ using Cartan connections. A generalisation of this approach to higher dimensions was mentioned in [Nur11]. The precise relation between the cited works and the construction here has been shown recently by Šilhan-Žádník, [ŠŽ]: It is based on an interpretation of the explicit formula for the Dunajski-Tod conformal metric in terms of 'Thomas's projective parameters', which in turn has relations to tractor calculus for projective structures and the projective ambient metric, [BEG94], and thus provides a link to the Fefferman-type interpretation of the construction.

Outlook. This construction leads to interesting questions for future work. In signature $(2, 2)$ Dunajski-Tod could show, [DT10], Theorem 4.1, that

one has a 1 : 1-correspondence between compatible (pseudo)-Riemannian metrics for the original projective class and (para)-Kähler-metrics in the induced conformal class. In forthcoming joint work with J. Šilhan and V. Žádník we will discuss this relation in terms of BGG-solutions to certain projective and conformal equations. Another problem that will be treated is to characterise the resulting conformal structures. As is shown in this article, the existence of a certain pure twistor spinor should play a large role in this, but additional data is necessary to characterise the structures precisely. It would also be interesting to study the ambient-metrics of the induced conformal Fefferman-spaces, as it was done for certain generic 2-distributions in [LN11].

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2. BASIC FACTS ABOUT PARABOLIC GEOMETRIES AND SOME BACKGROUND ON PROJECTIVE AND CONFORMAL STRUCTURES

2.1. Parabolic geometries. Let G be a Lie group with Lie algebra \mathfrak{g} and $P \subset G$ a closed subgroup with Lie algebra \mathfrak{p} . A *Cartan geometry* (\mathcal{G}, ω) of type (G, P) is a P -principal bundle $\mathcal{G} \rightarrow M$ together with a *Cartan connection* $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$, i.e., a \mathfrak{g} -valued 1-form on \mathcal{G} that i) is P -equivariant, ii) maps each fundamental vector field ζ_X to its generator $X \in \mathfrak{p}$, and iii) defines a linear isomorphism $\omega(u) : T_u\mathcal{G} \rightarrow \mathfrak{g}$ for each $u \in \mathcal{G}$.

The *curvature* of a Cartan connection ω is the 2-form $K \in \Omega^2(\mathcal{G}, \mathfrak{g})$ defined as

$$K(\xi, \eta) = d\omega(\xi, \eta) + [\omega(\xi), \omega(\eta)]$$

for $\xi, \eta \in \mathfrak{X}(\mathcal{G})$. It is equivalently encoded in the curvature function $\kappa : \mathcal{G} \rightarrow \Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$

$$\kappa(u)(X + \mathfrak{p}, Y + \mathfrak{p}) = K(\omega^{-1}(u)(X), \omega^{-1}(u)(Y)).$$

The curvature is a complete obstruction to local equivalence with the homogeneous model $G \rightarrow G/P$ endowed with the Maurer-Cartan form ω^{MC} . If the image of κ is contained in $\Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{p}$, then (\mathcal{G}, ω) is called *torsion-free*.

A *parabolic geometry* is a Cartan geometry of type (G, P) , where G is a semisimple Lie group and P is a parabolic subgroup. Every parabolic subgroup is the semidirect product $P = G_0 \ltimes P_+$ of a reductive Lie group G_0 and a normal subgroup $P_+ \subset P$. The Lie algebra \mathfrak{p}_+ is the orthogonal complement of \mathfrak{p} in \mathfrak{g} with respect to the Killing form, $P_+ = \exp(\mathfrak{p}_+)$ and $G_0 \cong P/P_+$. Since G_0 is reductive, its Lie algebra $\mathfrak{g}_0 = \mathfrak{g}_0^{ss} \oplus \mathfrak{z}(\mathfrak{g}_0)$ decomposes into the semisimple part $\mathfrak{g}_0^{ss} = [\mathfrak{g}_0, \mathfrak{g}_0]$ and the centre $\mathfrak{z}(\mathfrak{g}_0)$. For

parabolic geometries there is a natural choice of a normalisation condition, which reads $\partial^*(\kappa) = 0$, where

$$\partial^* : \Lambda^k(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g} \rightarrow \Lambda^{k-1}(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$$

is the *Kostant codifferential* [Kos61]. The *harmonic curvature* κ_H of a normal parabolic geometry is the image of κ under the projection $\ker \partial^* \rightarrow \ker \partial / \text{im} \partial^*$. The parabolic geometries we are mainly interested here (i.e. projective and conformal geometries) are automatically *regular*, see [ČS09], and in that case the entire curvature κ is completely determined by κ_H .

A technical tool that we will often employ are Weyl structures for parabolic geometries, cf. [ČS03, ČS09] for a detailed account. A *Weyl structure* of (\mathcal{G}, ω) is a reduction of structure group $j : \mathcal{G}_0 \hookrightarrow \mathcal{G}$ of the P -principal bundle \mathcal{G} to a G_0 -bundle \mathcal{G}_0 .

Every Cartan connection ω naturally extends to a principal bundle connection $\hat{\omega}$ on the G -principal bundle $\hat{\mathcal{G}} = \mathcal{G} \times_P G$. The principal bundle connection $\hat{\omega}$ induces a vector bundle connection $\nabla^\mathcal{V}$ on each associated bundle $\mathcal{V} = \mathcal{G} \times_P \mathbb{V} = \hat{\mathcal{G}} \times_G \mathbb{V}$ for a G -representation \mathbb{V} . Bundles \mathcal{V} and connections $\nabla^\mathcal{V}$ arising in this way are called *tractor bundles* and *tractor connections*. The tractor connections induced by normal Cartan connections for parabolic geometries are called normal tractor connections.

2.2. Normal solutions of first BGG-equations as parallel tractor sections. In [ČSS01], and later in a simplified manner in [CD01], it was shown that for a given tractor bundle \mathcal{V} one can associate a natural sequence of differential operators,

$$\mathcal{H}_0 \xrightarrow{\Theta_0^\mathcal{V}} \mathcal{H}_1 \xrightarrow{\Theta_1^\mathcal{V}} \dots \xrightarrow{\Theta_{n-1}^\mathcal{V}} \mathcal{H}_n.$$

The operators $\Theta_k^\mathcal{V}$ are the *BGG-operators*, which operate between natural sub-quotients \mathcal{H}_k of $\Omega^k(M, \mathcal{V})$. We remark that $\Theta_k^\mathcal{V}$ form a complex if and only if the geometry (\mathcal{G}, ω) is locally flat.

We won't discuss the general construction here, for which we refer to the articles mentioned above or [Ham09], and just state the basic properties of the *first BGG-operator* $\Theta_0^\mathcal{V} : \Gamma(\mathcal{H}_0) \rightarrow \Gamma(\mathcal{H}_1)$. The operator defines an overdetermined system of differential equations on $\sigma \in \Gamma(\mathcal{H}_0)$, $\Theta_0^\mathcal{V}(\sigma) \stackrel{!}{=} 0$, which is termed the *first BGG-equation*.

For the projective and conformal structures we discuss below, we will be able to encode a number of interesting geometric equations as first BGG-equations. In those cases solutions of the first BGG-equations are always in 1 : 1-correspondence with parallel sections of the defining tractor bundle \mathcal{V} , cf. [Ham09]. In general one only has 1 : 1-correspondence between parallel sections and a subspace of solutions of $\Theta_0^\mathcal{V}(\sigma) = 0$, which are called *normal solutions*. This correspondence is realised as follows: The bundle \mathcal{H}_0 is a natural quotient of \mathcal{V} , $\mathcal{V} \xrightarrow{\Pi_0} \mathcal{H}_0$, and the BGG-construction defines a natural differential splitting operator $\Gamma(\mathcal{H}_0) \xrightarrow{L_0^\mathcal{V}} \Gamma(\mathcal{V})$ of that projection. Then a solution of $\Theta_0^\mathcal{V}(\sigma) = 0$ is normal if and only if $\nabla^\mathcal{V} L^\mathcal{V}(\sigma) = 0$.

In the following we describe projective and conformal structures. To write down explicit formulas it will be useful to employ abstract index notation, cf. [PR87]: we write $\mathbf{E}_a = T^*M$, $\mathbf{E}^a = TM$ and multiple indices as in $\mathbf{E}_{ab} =$

$T^*M \otimes T^*M$ denote tensor products. Indices between squared brackets are skew, as in $\mathbf{E}_{[ab]} = \Lambda^2 T^*M$, and indices between round brackets are symmetric, as in $\mathbf{E}^{(ab)} = S^2 T^*M$.

2.3. Projective Structures. Let M be a manifold of dimension $n \geq 2$ endowed with a projective class of torsion-free affine connections $[D]$: two connections D and \hat{D} are projectively equivalent if they describe the same geodesics as unparameterised curves. This is the case if and only if there is a $\Upsilon_a \in \mathbf{E}_a$ such that for all $\xi^a \in \mathbf{E}^a$,

$$\hat{D}_a \xi^b = D_a \xi^b + \Upsilon_a \xi^b + \Upsilon_p \xi^p \delta_a^b, \quad (1)$$

where $\delta = \text{id}_{TM}$ is the Kronecker-symbol for the identity on TM , cf. e.g. [EM07] and [BEG94].

Let R be the curvature of D . With the Schouten tensor $P \in \mathbf{E}_{(ab)}$,

$$P_{ab} = \frac{1}{n-1} R_{pa}{}^p{}_b \quad (2)$$

one has the projective Weyl- and Cotton tensor

$$C_{c_1 c_2}{}^a{}_p = R_{c_1 c_2}{}^a{}_p + P_{c_1 p} \delta_{c_2}^a - P_{c_2 p} \delta_{c_1}^a, \quad (3)$$

$$A_{ac_1 c_2} = 2D_{[c_1} P_{c_2]a}. \quad (4)$$

An oriented projective structure $(M, [D])$ is equivalently encoded in a normal parabolic geometry of type $(SL(n+1), P)$, where P is the stabiliser of a ray in the standard representation \mathbb{R}^{n+1} . This classical result goes back to É.Cartan, [Car24]. For a modern treatment we refer to [Sha97, ČS09].

The parabolic subgroup $P \subset G$ is a semidirect product $P = GL(n) \ltimes (\mathbb{R}^n)^*$. The 1-dimensional representation of P

$$GL(n) \ltimes (\mathbb{R}^n)^* \rightarrow \mathbb{R}_+, (C, X) \mapsto \det(C)^{w \frac{n}{n+1}}$$

is denoted by $\mathbb{R}[w]$: the associated space $\mathbf{E}[w] := \mathcal{G} \times_P \mathbb{R}[w]$ are projective w -densities, which are just usual densities with a suitable parametrisation. If V is a P -representation and $\mathcal{V} = \mathcal{G} \times_P V$ its associated bundle we will simply write $V[w] = V \otimes \mathbb{R}[w]$ resp. $\mathcal{V}[w] = \mathcal{V} \otimes \mathbf{E}[w]$ for the weighted versions of the modelling representation resp. the corresponding associated bundles.

For the projective geometry $(M, [D])$ any choice of affine connection $D \in [D]$ yields a projective Weyl structure, and in particular the structure group of any tractor bundle is reduced to $G_0 = SL(n)$.

2.3.1. The projective standard tractor bundle. This is the associated bundle $\mathcal{T} = \mathcal{G} \times_P \mathbb{R}^{n+1}$. With respect to a choice of $D \in [D]$ we have $[\mathcal{T}]_D =$

$\begin{pmatrix} \mathbf{E}[-1] \\ \mathbf{E}^a[-1] \end{pmatrix}$, and $\Pi_0 : \mathcal{T} \rightarrow \mathbf{E}^a[-1] = \mathcal{H}_0^\mathcal{T}$ is the projectively invariant projection to the lowest slot. The tractor connection is given by $\nabla_c^\mathcal{T} \begin{pmatrix} \rho \\ \sigma^a \end{pmatrix} =$

$\begin{pmatrix} D_c \rho - P_{cp} \sigma^p \\ D_c \sigma^a + \rho \delta_c^a \end{pmatrix}$. The BGG-splitting operator is

$$L_0^\mathcal{T} : \mathbf{E}^a[-1] \rightarrow \mathcal{S}, \sigma^a \mapsto \begin{pmatrix} -\frac{1}{n} D_p \sigma^p \\ \sigma^a \end{pmatrix}$$

and the first BGG-operator of \mathcal{T} is

$$\Theta_0^{\mathcal{T}} : \mathbf{E}^a[-1] \rightarrow \mathbf{E}_{0c}^a[-1], \sigma^a \mapsto D_c \sigma^a - \frac{1}{n} \delta_c^a D_p \sigma^p. \quad (5)$$

Thus, $\ker \Theta_0^{\mathcal{T}}$ consists of vector fields which are mapped to multiples of the identity by D .

2.3.2. The projective dual standard tractor bundle. The dual bundle to \mathcal{T} is $\mathcal{T}^* = \mathcal{G} \times_P \mathbb{R}^{n+1*}$. Its decomposition under $D \in [D]$ is $[\mathcal{T}^*]_D = \begin{pmatrix} \mathbf{E}_a[1] \\ \mathbf{E}[1] \end{pmatrix}$ and $\Pi_0 : \mathcal{T}^* \rightarrow \mathbf{E}[1] = \mathcal{H}_0^{\mathcal{T}^*}$ is the projectively invariant projection to the lowest slot. The tractor connection is $\nabla_c^{\mathcal{T}^*} \begin{pmatrix} \varphi_a \\ \sigma \end{pmatrix} = \begin{pmatrix} D_c \varphi_a + P_{ca} \sigma \\ D_c \sigma - \varphi_c \end{pmatrix}$. The first splitting operator of \mathcal{T}^* is

$$L_0^{\mathcal{T}^*} : \mathbf{E}[1] \rightarrow \mathcal{T}^*, \sigma \mapsto \begin{pmatrix} D_a \sigma \\ \sigma \end{pmatrix}$$

and the first BGG-operator is

$$\Theta_0^{\mathcal{T}^*} : \mathbf{E}[1] \rightarrow \mathbf{E}_{(ab)}[1], \sigma \mapsto D_a D_b \sigma + \sigma P_{ab}. \quad (6)$$

Let $\sigma \in C^\infty(M)$ be a solution of $\Theta_0^{\mathcal{T}^*}(\sigma) = 0$ and define $\Upsilon_a = D_a(\log \frac{1}{|\sigma|})$. Then Υ is a well-defined 1-form on $U := M \setminus \sigma^{-1}(\{0\})$, and one can form the connection $\hat{D}, (1)$, that is projectively equivalent to the restriction of D to U . Then (6) implies, cf. [Ham09, ČGH10], that $\text{Ric}(\hat{D}) = 0$. We will call any connection \hat{D} that is defined on an open-dense subset U of M and is contained in the restriction of $[D]$ to U an *almost Ricci-flat structure* of $[D]$, or $\hat{D} \in \mathbf{aRs}([D])$. It will sometimes be useful to regard $\mathbf{aRs}([D]) \subset [D]$, even though the almost Ricci-flat structures only give connections on an open-dense subset of M . Then, cf. [Ham09, ČGH10],

$$\mathbf{aRs}([D]) \cong \ker \Theta_0^{\mathcal{T}^*}. \quad (7)$$

2.4. Conformal spin structures. A *conformal structure* of signature (n, n) on an $n = p + q$ -dimensional manifold M is an equivalence class \mathcal{C} of pseudo-Riemannian metrics with two metrics g and \hat{g} being equivalent if $\hat{g} = e^{2f}g$ for a function $f \in C^\infty(M)$. Suppose we have a manifold with a conformal structure of signature (n, n) . Let \mathcal{G}_0 be the associated conformal frame bundle with structure group the conformal group $\text{CO}_o(n, n) = \mathbb{R}_+ \times \text{SO}_o(n, n)$ preserving both orientations. Then a *conformal spin structure* on M is a reduction of structure group of \mathcal{G}_0 to $\text{CSpin}(n, n) = \mathbb{R}_+ \times \text{Spin}(n, n)$. As for projective structures, it is useful to employ a suitable parametrisation of densities: the *conformal density bundles* $\mathbf{E}[w]$, which are the line bundles associated to the 1-dimensional representations $(c, C) \mapsto c^r \in \mathbb{R}_+$ of $\text{CSpin}(n, n) = \mathbb{R}_+ \times \text{Spin}(n, n)$.

Let us now briefly introduce the main curvature quantities of the conformal structure \mathcal{C} , cf. e.g. [Eas96]. For $g \in \mathcal{C}$, let, with $m = 2n$,

$$P = P(g) := \frac{1}{m-2}(\text{Ric}(g) - \frac{\text{Sc}(g)}{2(m-1)}g)$$

be the *Schouten tensor*; this is a trace modification of the Ricci curvature $\text{Ric}(g)$ by a multiple of the scalar curvature $\text{Sc}(g)$. The trace of the Schouten tensor is denoted $J = g^{pq} P_{pq}$.

It is well known that (since we always have dimension $\geq 4 > 3$), the complete obstruction against conformal flatness of (M, \mathcal{C}) is the *Weyl curvature*

$$C_{ab\ d}^{\ c} := R_{ab\ d}^{\ c} - 2\delta_{[a}^c P_{b]d} + 2g_{d[a} P_{b]}^c,$$

where indices between square brackets are skewed over.

A conformal spin structures of signature (n, n) is equivalently encoded in a normal parabolic geometry of type $(\text{Spin}(n+1, n+1), \tilde{P})$, where \tilde{P} is the stabiliser of a ray in $\mathbb{R}^{n+1, n+1}$. Any choice of $g \in \mathcal{C}$ yields a Weyl structure of (\mathcal{G}, ω) , and this reduces the structure group of a tractor bundle to $\tilde{G}_0 = \text{Spin}(n, n)$.

2.4.1. The conformal standard tractor bundle. This is the associated bundle $\tilde{\mathcal{T}} = \tilde{\mathcal{G}} \times_{\tilde{P}} \mathbb{R}^{n+1, n+1}$, and with respect to $g \in \mathcal{C}$ it decomposes $[\tilde{\mathcal{T}}]_g = \begin{pmatrix} \mathbf{E}[-1] \\ \mathbf{E}_a[1] \\ \mathbf{E}[1] \end{pmatrix}$, and $\Pi_0 : \tilde{\mathcal{T}} \rightarrow \mathbf{E}[1] = \mathcal{H}_0^{\tilde{\mathcal{T}}}$ is the projectively invariant projection

to the lowest slot. $\tilde{\mathcal{T}}$ carries invariant tractor metric $[\mathbf{h}]_{\mathbf{g}} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \mathbf{g} & 0 \\ 1 & 0 & 0 \end{pmatrix}$,

which is compatible with the standard tractor connection $[\nabla_c^{\tilde{\mathcal{T}}} \begin{pmatrix} \rho \\ \varphi_a \\ \sigma \end{pmatrix}]_g =$

$\begin{pmatrix} D_c \rho - P_c^b \varphi_b \\ D_c \varphi_a + \sigma P_{ca} + \rho g_{ca} \\ D_c \sigma - \varphi_c \end{pmatrix}$. The BGG-splitting operator of $\tilde{\mathcal{T}}$ is

$$L_0^{\tilde{\mathcal{T}}} : \mathbf{E}[1] \rightarrow \tilde{\mathcal{T}}, \quad \sigma \mapsto \begin{pmatrix} \frac{1}{2n}(\Delta - J)\sigma \\ D\sigma \\ \sigma \end{pmatrix} \quad (8)$$

with the convention $\Delta = -D^p D_p$. The first BGG-operator is

$$\Theta_0^{\tilde{\mathcal{T}}} : \mathbf{E}[1] \rightarrow \mathbf{E}_{0(ab)}, \quad \sigma \mapsto (D_a D_b \sigma + P_{ab} \sigma)_0. \quad (9)$$

It is well known that

$$(D_a D_b \sigma + P_{ab} \sigma)_0 = 0 \Leftrightarrow \sigma^{-2} \mathbf{g} \text{ is Einstein on } U, \quad (10)$$

and we call the set of solutions of (10) the space of *almost Einstein structures* of \mathcal{C} , cf. [Gov10], i.e.:

$$\mathbf{aEs}(\mathcal{C}) = \ker \Theta_0^{\tilde{\mathcal{T}}} \subset \mathbf{E}[1]. \quad (11)$$

It will sometimes be convenient to regard $\mathbf{aEs}(\mathcal{C}) \subset \mathcal{C}$, even if these Einstein-metrics are only defined on an open-dense subset.

2.4.2. *The spin tractor bundle.* Since \mathcal{C} is a conformal spin structure and modelled on a Cartan geometry of type $(\text{Spin}(n+1, n+1), \tilde{P})$ we can define the spin tractor bundle as $\tilde{\mathcal{S}} = \tilde{\mathcal{G}} \times_{\tilde{P}} \Delta^{n+1, n+1}$. Since we work in even signature, this decomposes into $\tilde{\mathcal{S}}_{\pm} = \tilde{\mathcal{G}} \times_{\tilde{P}} \Delta_{\pm}^{n+1, n+1}$. Under a choice of $g \in \mathcal{C}$ the spin tractor bundles decompose as follows: $[\tilde{\mathcal{S}}_{\pm}]_g = \begin{pmatrix} S_{\mp}[-\frac{1}{2}] \\ S_{\pm}[\frac{1}{2}] \end{pmatrix}$.

$\Pi_0 : \tilde{\mathcal{S}}_{\pm} \rightarrow S_{\pm}[\frac{1}{2}] = \mathcal{H}_0^{\tilde{\mathcal{S}}_{\pm}}$ is the projectively invariant projection to the lowest slot. The Clifford action of the conformal standard tractor bundle $\tilde{\mathcal{T}}$ on $\tilde{\mathcal{S}}$ is given by

$$\begin{pmatrix} \rho \\ \varphi_a \\ \sigma \end{pmatrix} \cdot \begin{pmatrix} \tau \\ \chi \end{pmatrix} = \begin{pmatrix} -\varphi_a \cdot \tau + \sqrt{2}\rho\chi \\ \varphi_a \cdot \chi - \sqrt{2}\sigma\tau \end{pmatrix}, \quad (12)$$

cf. [Ham09, Ham10]. $\tilde{\mathcal{S}} = \tilde{\mathcal{S}}_+ \oplus \tilde{\mathcal{S}}_-$ carries the spin tractor connections that is induced from the standard tractor connection on $\tilde{\mathcal{T}}$: $[\nabla_c^{\tilde{\mathcal{S}}} \begin{pmatrix} \tau \\ \chi \end{pmatrix}]_g = \begin{pmatrix} D_c\tau + \frac{1}{\sqrt{2}}P_{cp}\gamma^p\chi \\ D_c\chi + \frac{1}{\sqrt{2}}\gamma_c\tau \end{pmatrix}$.

The BGG-splitting operator of $\tilde{\mathcal{S}}_{\pm}$ is

$$L_0^{\tilde{\mathcal{S}}_{\pm}} : \Gamma(S_{\pm}[\frac{1}{2}]) \rightarrow \Gamma(\tilde{\mathcal{S}}_{\pm}), \quad \chi \mapsto \begin{pmatrix} \frac{1}{\sqrt{2n}}\mathcal{D}\chi \\ \chi \end{pmatrix}. \quad (13)$$

Here

$$\mathcal{D} : \Gamma(S_{\pm}) \rightarrow \Gamma(S_{\mp}), \quad \mathcal{D} := \gamma^p D_p,$$

is the *Dirac* operator. The first BGG-operator is

$$\begin{aligned} \Theta_0^{\tilde{\mathcal{S}}} : \Gamma(S_{\pm}[\frac{1}{2}]) &\rightarrow \Gamma(T^*M \otimes S_{\pm}[\frac{1}{2}]), \\ \Theta_0^{\tilde{\mathcal{S}}}(\chi) &:= D\chi + \frac{1}{2n}\gamma\mathcal{D}\chi. \end{aligned}$$

This is the *twistor operator* (cf. e.g. [BFGK90]), which is alternatively described as the projection of the Levi-Civita derivative of a spinor to the kernel of Clifford multiplication. The kernel of the twistor operator is called the space of twistor spinors $\mathbf{Tw}(\mathcal{C})$, and Π_0 induces an isomorphism of the space of $\nabla^{\tilde{\mathcal{S}}}$ -parallel sections of $\tilde{\mathcal{S}}$ with $\mathbf{Tw}(\mathcal{C})$ in $\Gamma(S[\frac{1}{2}])$.

2.4.3. *Conformal holonomy.* The *conformal holonomy* of a conformal spin structure \mathcal{C} is defined as

$$\text{Hol}(\mathcal{C}) := \text{Hol}(\nabla^{\tilde{\mathcal{T}}}) = \text{Hol}(\nabla^{\tilde{\mathcal{S}}}) \subset \text{Spin}(p+1, q+1). \quad (14)$$

3. FEFFERMAN-TYPE CONSTRUCTIONS

Let \tilde{G} be a Lie group with Lie algebra $\mathfrak{so}(p+1, q+1)$ and let $\tilde{P} \subset \tilde{G}$ be the stabiliser of a null-line $\ell \subset \mathbb{R}^{p+1, q+1}$. Suppose we have an inclusion of Lie groups $i : G \hookrightarrow \tilde{G}$ with derivative $i : \mathfrak{g} \rightarrow \tilde{\mathfrak{g}}$. Assume that the G -orbit $G \cdot o$ is open in \tilde{G}/\tilde{P} and let $P \subset G$ be a parabolic subgroup that contains the intersection $Q = G \cap \tilde{P}$. (In particular, this implies $\mathfrak{g}/\mathfrak{p} \cong \tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}$ and

$\tilde{\mathfrak{g}} = \mathfrak{g} + \tilde{\mathfrak{p}}$). This is the algebraic set up for Fefferman-type constructions as in [Čap06] inducing conformal structures of signature (p, q) .

Since Fefferman-type constructions have been studied quite intensively in the literature already, we recall the general construction here only briefly and refer to the literature (e.g. [ČG08] and [ČS09]) for details. Let $(\mathcal{G} \rightarrow M, \omega)$ be a parabolic geometry of type (G, P) . One can form the correspondence space $\tilde{M} = \mathcal{G}/Q = \mathcal{G} \times_P P/Q$. The projection $\mathcal{G} \rightarrow \tilde{M}$ is a Q -principal bundle, and from the defining properties of a Cartan connection one sees that $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ is a Cartan connection also on $\mathcal{G} \rightarrow \tilde{M}$. So $(\mathcal{G} \rightarrow \tilde{M}, \omega)$ is a Cartan geometry of type (G, Q) . As a next step, one considers the extended bundle $\tilde{\mathcal{G}} = \mathcal{G} \times_Q \tilde{P}$ with respect to the inclusion $Q \hookrightarrow \tilde{P}$. This is a principal bundle over \tilde{M} with structure group \tilde{P} . Equivariant extension of ω yields a unique Cartan connection $\tilde{\omega} \in \Omega^1(\tilde{\mathcal{G}}, \tilde{\mathfrak{g}})$ that restricts to ω on \mathcal{G} . Thus, one obtains a functor from parabolic geometries $(\mathcal{G} \rightarrow M, \omega)$ of type (G, P) to parabolic geometries $(\tilde{\mathcal{G}} \rightarrow \tilde{M}, \tilde{\omega})$ of type (\tilde{G}, \tilde{P}) .

3.1. Normality. Next we derive a criterion suitable for our purposes that tells when this Fefferman-construction assigns a *normal* conformal geometry $(\tilde{\mathcal{G}}, \tilde{\omega})$ to a regular, normal parabolic geometry (\mathcal{G}, ω) . We will throughout assume that the restriction of the Killing form \tilde{B} of $\tilde{\mathfrak{g}}$ to \mathfrak{g} is a non-zero multiple of the Killing form B of \mathfrak{g} (which is true for the inclusions we are interested in). We use \tilde{B} to identify $(\mathfrak{g}/\mathfrak{p})^* \cong \mathfrak{p}_+$ and $(\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \cong \tilde{\mathfrak{p}}_+$. Let $X_1, \dots, X_n \in \mathfrak{g}$ be elements inducing a basis of $\mathfrak{g}/\mathfrak{p}$ and extend these elements by $X_{n+1}, \dots, X_m \in \mathfrak{p}$ such that X_1, \dots, X_m induce a basis of $\mathfrak{g}/\mathfrak{q} \cong \tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}$. Let Z_1, \dots, Z_n be the dual basis of X_1, \dots, X_n in $(\mathfrak{g}/\mathfrak{p})^* \cong \mathfrak{p}_+$ and $\tilde{Z}_1, \dots, \tilde{Z}_m$ be the dual basis of X_1, \dots, X_m in $(\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \cong \tilde{\mathfrak{p}}_+$. Then $\tilde{Z}_j - Z_j$ for $j = 1, \dots, n$ are contained in the orthogonal complement $\mathfrak{g}^\perp \subset \tilde{\mathfrak{g}}$ with respect to the Killing form: For $i = 1, \dots, n$, we have

$$\tilde{B}(X_i, \tilde{Z}_j - Z_j) = \tilde{B}(X_i, \tilde{Z}_j) - \tilde{B}(X_i, Z_j) = \delta_{i,j} - \delta_{i,j} = 0.$$

For $i = n+1, \dots, m$, we have $\tilde{B}(X_i, \tilde{Z}_j) = 0$ since $i \neq j$ and $\tilde{B}(X_i, Z_j) = 0$ since $X_i \in \mathfrak{p}$ and $Z_j \in \mathfrak{p}_+$. Finally, we have $\tilde{B}(\mathfrak{q}, \tilde{Z}_j) = 0$ since $\mathfrak{q} \subset \tilde{\mathfrak{p}}$ and $\tilde{Z}_j \in \tilde{\mathfrak{p}}_+$ and $\tilde{B}(\mathfrak{q}, Z_j) = 0$ since $\mathfrak{q} \subset \mathfrak{p}$ and $Z_j \in \mathfrak{p}_+$.

Now suppose $\kappa : \mathcal{G} \rightarrow \Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$ is the curvature function of a normal parabolic geometry of type (G, P) . The normality condition reads

$$\begin{aligned} \partial^*(\kappa)(u)(X) &= \partial_1^*(\kappa)(u)(X) + \partial_2^*(\kappa)(u)(X) \\ &= 2 \sum_{i=1}^n [\kappa(u)(X_i, X), Z_i] + \sum_{i=1}^n \kappa(u)([X_i, [Z_i, X]]) = 0 \end{aligned} \quad (15)$$

for all $u \in \mathcal{G}$ and $X \in \mathfrak{g}$. Let $\tilde{\kappa} : \tilde{\mathcal{G}} \rightarrow \Lambda^2(\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \mathfrak{g}$ be the curvature function of the associated conformal geometry. This geometry is normal if and only if

$$\tilde{\partial}^* \tilde{\kappa}(\tilde{u})(\tilde{X}) = 2 \sum_{i=1}^m [\tilde{\kappa}(\tilde{u})(X_i, \tilde{X}), \tilde{Z}_i] = 0 \quad (16)$$

for all $\tilde{u} \in \tilde{\mathcal{G}}$ and $\tilde{X} \in \tilde{\mathfrak{g}}$. By construction, we know that $\tilde{\kappa}$ is a \tilde{P} -equivariant extension of κ and elements of \mathfrak{p} insert trivially into $\tilde{\kappa}$. Since also $\tilde{\partial}^*$ is \tilde{P} -equivariant, to prove normality of $\tilde{\kappa}$ it suffices to verify that

$$\tilde{\partial}^* \tilde{\kappa}(u)(X) = 2 \sum_{i=1}^n [\tilde{\kappa}(u)(X_i, X), \tilde{Z}_i] = 2 \sum_{i=1}^n [\kappa(u)(X_i, X), \tilde{Z}_i] = 0 \quad (17)$$

for all $u \in \mathcal{G}$ and $X \in \mathfrak{g}$.

Proposition 3.1. *Suppose that the parabolic geometry (\mathcal{G}, ω) of type (G, P) is regular and normal, the curvature function κ takes values in $\Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes (\mathfrak{g} \cap \tilde{\mathfrak{p}})$ and the two summands in the normality condition vanish separately, i.e. $\partial_1^*(\kappa) = \partial_2^*(\kappa) = 0$. Then $\tilde{\partial}^*(\tilde{\kappa}) = 0$, i.e. the induced conformal parabolic geometry is normal.*

Proof. Using that $\partial_1^*(\kappa)(u)(X) = 2 \sum_{i=1}^n [\kappa(u)(X_i, X), Z_i] = 0$ and (17), we can rewrite $\tilde{\partial}^* \tilde{\kappa}(u)(X)$ as

$$2 \sum_{i=1}^n [\kappa(u)(X_i, X), \tilde{Z}_i - Z_i]. \quad (18)$$

We have observed that $\tilde{Z}_i - Z_i \in \mathfrak{g}^\perp$ and by construction $\kappa(u)(X_i, X) \in \mathfrak{g}$. Since the decomposition $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{g}^\perp$ is invariant under the action of \mathfrak{g} , this implies that $\tilde{\partial}^* \tilde{\kappa}(u)(X) = \sum_{i=1}^n [\kappa(u)(X_i, X), \tilde{Z}_i - Z_i] \in \mathfrak{g}^\perp$. On the other hand, since by assumption $\tilde{\kappa}(u)(X_i, X) \in \tilde{\mathfrak{p}}$ and $\tilde{Z}_i \in \tilde{\mathfrak{p}}_+$, we have $\tilde{\partial}^* \tilde{\kappa}(u)(X) \in \tilde{\mathfrak{p}}_+$. But the intersection $\mathfrak{g}^\perp \cap \tilde{\mathfrak{p}}_+$ is zero: Note that $\tilde{\mathfrak{p}}_+ = \tilde{\mathfrak{p}}^\perp$, so any element in $\mathfrak{g}^\perp \cap \tilde{\mathfrak{p}}_+$ is orthogonal to $\mathfrak{g} + \tilde{\mathfrak{p}} = \tilde{\mathfrak{g}}$. Since the Killing form is non-degenerate this implies $\mathfrak{g}^\perp \cap \tilde{\mathfrak{p}}_+ = 0$ and we conclude that $\tilde{\partial}^* \tilde{\kappa} = 0$. \square

Remark 3.1. Suppose κ is torsion-free, then Corollary 3.2 in [Čap05] shows that it suffices to check that both ∂_1^* and ∂_2^* annihilate the harmonic curvature to conclude that they annihilate κ . If there is only one harmonic curvature component, then always $\partial_1^*(\kappa_H) = \partial_2^*(\kappa_H) = 0$. The reason for this is that the two summands $\partial_1^*(\kappa_H)(u)(X)$ and $\partial_2^*(\kappa_H)(u)(X)$ are contained in different grading components and cannot cancel.

4. FROM PROJECTIVE TO CONFORMAL STRUCTURES OF SIGNATURE (n, n)

4.1. The construction. For this construction denote by $\Delta = \Delta_+^{n+1, n+1} \oplus \Delta_-^{n+1, n+1}$ the real 2^{n+1} -dimensional spin representation of $\tilde{G} = Spin(n+1, n+1)$. Then we fix two pure spinors $s_F \in \Delta_-^{n+1, n+1}$, $s_E \in \Delta_\pm^{n+1, n+1}$ with non-trivial pairing - here s_E lies in $\Delta_+^{n+1, n+1}$ if n is even or $\Delta_-^{n+1, n+1}$ if n is odd, cf. [Bau81]. These assumptions guarantee that the kernels $E, F \subset \mathbb{R}^{n+1, n+1}$ of s_E, s_F with respect to Clifford multiplication are complementary maximally isotropic subspaces. Let now

$$G := \{g \in Spin(n+1, n+1) : g \cdot s_E = s_E, g \cdot s_F = s_F\} \cong SL(n+1),$$

and this defines an embedding

$$SL(n+1) \xrightarrow{i} Spin(n+1, n+1).$$

Under $\mathrm{SL}(n+1)$ the space $\mathbb{R}^{n+1,n+1}$ then decomposes into a copy of the standard representation and the dual representation:

$$\mathbb{R}^{n+1,n+1} = E \oplus F = \mathbb{R}^{n+1} \oplus \mathbb{R}^{n+1*}. \quad (19)$$

Note that this decomposition determines a G -invariant skew-symmetric involution $\mathbb{K} \in \Lambda^2 \mathbb{R}^{n+1}$ acting by the identity on E and minus the identity on F . In particular an embedding of $\mathrm{SL}(n+1)$ can also be defined via such an involution.

We will realise $\mathrm{Spin}(n+1, n+1)$ with respect to the split signature form

$$h = \begin{pmatrix} 0 & I_{n+1} \\ I_{n+1} & 0 \end{pmatrix}, \quad (20)$$

such that the corresponding inclusion on the Lie algebra level is given by

$$\begin{aligned} \mathfrak{sl}(n+1) &\hookrightarrow \mathfrak{so}(n+1, n+1) \\ A &\mapsto \begin{pmatrix} A & 0 \\ 0 & -(A^t) \end{pmatrix}. \end{aligned} \quad (21)$$

Let $\tilde{P} \subset \tilde{G}$ be the stabiliser of the ray $\mathbb{R}_+ \tilde{v}_+$ through the null-vector

$$\tilde{v}_+ = (1 \ 0 \ \cdots \ 0 \ 1)^t \in \mathbb{R}^{n+1,n+1}. \quad (22)$$

Then the group $Q := i^{-1}(\tilde{P}) \subset G$ consists of matrices of the form

$$\begin{pmatrix} a & Z^t & b \\ 0 & A & Y \\ 0 & 0 & a^{-1} \end{pmatrix}, \quad (23)$$

with $a \in \mathbb{R}_+$, $b \in \mathbb{R}$, $Z, Y \in \mathbb{R}^{n-1}$ and $A \in \mathrm{SL}(n-1)$. This group Q , which is not a parabolic subgroup, is contained in the parabolic subgroup $P \subset G$, of the form

$$\begin{pmatrix} a & Z^t & b \\ 0 & A & Y \\ 0 & X^t & c \end{pmatrix}, \quad (24)$$

defined as the stabilizer in G of the ray $\mathbb{R}_+ v_+$ through the vector

$$v_+ = (1 \ 0 \ \cdots \ 0)^t \in \mathbb{R}^{n+1}. \quad (25)$$

We denote by $\tilde{\mathfrak{g}}, \tilde{\mathfrak{p}}, \mathfrak{g}, \mathfrak{p}, \mathfrak{q}$ the Lie algebras of the groups introduced above. Dimension count shows that the derivative $i' : \mathfrak{g} \rightarrow \tilde{\mathfrak{g}}$ of the inclusion $i : G \hookrightarrow \tilde{G}$ induces an isomorphism $\mathfrak{g}/\mathfrak{q} \cong \tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}$. Hence the orbit $G \cdot o \subset \tilde{G}/\tilde{P}$ is open. (But the action of G on \tilde{G}/\tilde{P} is not transitive; in addition to the open orbit there are two lower dimensional orbits.) That means that we can perform a Fefferman-type construction (as explained in 3) from parabolic geometries of type (G, P) on to parabolic geometries of type (\tilde{G}, \tilde{P}) . Since every parabolic geometry of type (\tilde{G}, \tilde{P}) determines an underlying conformal spin structure (see e.g. [ČS09]), this yields a construction of a conformal spin structure on the correspondence space \tilde{M} over a projective manifold M .

Let us describe the correspondence space $\tilde{M} = \mathcal{G} \times_P P/Q$ more carefully. Via the Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$, the cotangent bundle T^*M can be

identified with $\mathcal{G} \times_P (\mathfrak{g}/\mathfrak{p})^*$. Consider an arbitrary element $p \in P$; it is of the form

$$p = \begin{pmatrix} a & Y^t \\ 0 & A \end{pmatrix} \quad (26)$$

for some $A \in \mathrm{GL}^+(n)$, $a = (\det A)^{-1}$ and $Y \in \mathbb{R}^n$. The P -representation on $(\mathfrak{g}/\mathfrak{p})^* \cong \mathbb{R}^n$ is given by $\rho(p)(Y) = a(A^{-1})^t Y$. If we form the tensor product of this representation on $(\mathfrak{g}/\mathfrak{p})^*$ with the 1-dimensional representation given by $\rho(p) = (\det A)^2 = (a^{-1})^2$, then the resulting representation is $\rho(p)(Y) = a^{-1}(A^{-1})^t Y$. The corresponding representation space shall be denoted by $(\mathfrak{g}/\mathfrak{p})^*[2]$. The action defined by this representation is transitive on $(\mathfrak{g}/\mathfrak{p})^*[2] \setminus \{0\}$, and the isotropy subgroup of $e_n \in \mathbb{R}^n \cong (\mathfrak{g}/\mathfrak{p})^*[2] \setminus \{0\}$ is the group Q . Thus we may identify the correspondence space \tilde{M} with $T^*M[2] \setminus \{0\}$.

Proposition 4.1. *The Fefferman-type construction for the pairs of Lie groups (G, P) and (\tilde{G}, \tilde{P}) as above naturally associates a conformal spin structure of signature (n, n) on $\tilde{M} = T^*M[2] \setminus \{0\}$ to an n -dimensional projective structure on M .*

4.2. Induced structures on the conformal Fefferman space. Let $L = \mathbb{R}\tilde{v}_+$ be the line spanned by the null-vector \tilde{v}_+ and let L^\perp be the orthogonal complement with respect to h . Consider $\bar{E} = E \cap L^\perp$ and $\bar{F} = F \cap L^\perp$. The line L is neither contained in \bar{E} nor \bar{F} , and these two subspaces induce n -dimensional isotropic subspaces e, f in L^\perp/L with 1-dimensional intersection k .

We have a \mathfrak{q} -invariant identification $\mathfrak{g}/\mathfrak{q} \cong L^* \otimes L^\perp/L$ via $X \mapsto (\tilde{v}_+)^* \otimes X \cdot \tilde{v}_+$. Under this identification the subspace $f = \mathfrak{p}/\mathfrak{q} \subset \mathfrak{g}/\mathfrak{q}$ corresponds to $L^* \otimes f$. We denote by $e \subset \mathfrak{g}/\mathfrak{q}$ the subspace corresponding to $L^* \otimes e$, and then $e \cap f = \mathfrak{p}'/\mathfrak{q} \subset \mathfrak{g}/\mathfrak{q}$, where \mathfrak{p}' is the Lie algebra of P' as in (32), which corresponds to $L^* \otimes k$. The G -invariant involution $\mathbb{K} \in \mathfrak{so}(n+1, n+1) = \tilde{\mathfrak{g}}$ defines a Q -invariant element $k := \mathbb{K}/\tilde{\mathfrak{p}} \in \tilde{\mathfrak{g}}/\tilde{\mathfrak{p}} \cong \mathfrak{g}/\mathfrak{q}$, which spans the 1-dimensional intersection $e \cap f$. The sum $e + f$ coincides with the orthogonal complement of k in $\mathfrak{g}/\mathfrak{q}$. Thus

$$k \in e \cap f \subset k^\perp = e + f \subset \mathfrak{g}/\mathfrak{q}, \quad (27)$$

both e and f are maximally isotropic (of dimension n) in $\mathfrak{g}/\mathfrak{q}$, and in particular k is null.

It follows, that the tangent bundle $T\tilde{M} = \mathcal{G} \times_Q \mathfrak{g}/\mathfrak{q}$ has two n -dimensional isotropic subbundles with one-dimensional intersection, corresponding to e and f and $e \cap f$. The bundle $\mathcal{G} \times_Q f$ is the vertical bundle for the projection $\tilde{M} \rightarrow M$.

The geometric tractor objects corresponding to the G -invariant algebraic data introduced in the beginning of 4.1 will be denoted as follows: The conformal standard tractor bundle $\tilde{\mathcal{T}} = \tilde{\mathcal{G}} \times_{\tilde{P}} \mathbb{R}^{n+1, n+1} = \mathcal{G} \times_Q \mathbb{R}^{n+1, n+1}$ naturally decomposes as

$$\tilde{\mathcal{T}} = \tilde{\mathcal{E}} \oplus \tilde{\mathcal{F}}. \quad (28)$$

The involution \mathbb{K} gives rise to an adjoint tractor $\mathbf{K} \in \Gamma(\Lambda^2 \tilde{\mathcal{T}})$ and the invariant spinors give rise to (pure) spin tractors $\mathbf{s}_E \in \Gamma(\tilde{\mathcal{S}}_{\pm}) = \Gamma(\mathcal{G} \times_Q \Delta_{\pm})$ and $\mathbf{s}_F \in \Gamma(\tilde{\mathcal{S}}_-) = \Gamma(\mathcal{G} \times_Q \Delta_-)$ with non-trivial pairing, cf. 4.1.

The conformal Cartan connection $\tilde{\omega} \in \Omega^1(\tilde{\mathcal{G}}, \mathfrak{g})$ obtained via the Fefferman construction induces a tractor connection $\nabla^{\tilde{\mathcal{V}}}$ on each conformal tractor bundle $\tilde{\mathcal{V}}$. By construction, the decomposition of the tractor bundle (28) is preserved by the induced conformal tractor connection and the adjoint tractor \mathbf{K} and the spin tractors $\mathbf{s}_E, \mathbf{s}_F$ are all parallel with respect to the induced tractor connections on the respective bundles. Note that we have not made any claims yet as to whether the additional structure on the conformal tractor bundles is preserved by the normal conformal tractor connection $\nabla^{\tilde{\mathcal{V}}, \text{nor}}$, which is a priori different from $\nabla^{\tilde{\mathcal{V}}}$.

4.3. Relation between projective and conformal parallel tractors.

Suppose \mathbb{V} is a \tilde{G} representation, which is then also a G representation, since $G \subset \tilde{G}$. Let $\mathcal{V} = \mathcal{G} \times_P \mathbb{V} \rightarrow M$ be the associated projective tractor bundle and let $\tilde{\mathcal{V}} = \tilde{\mathcal{G}} \times_{\tilde{P}} \mathbb{V} = \mathcal{G} \times_Q \mathbb{V} \rightarrow \tilde{M}$ be the associated conformal tractor bundle. Let $\nabla^{\mathcal{V}}$ and $\nabla^{\tilde{\mathcal{V}}}$ be the tractor connections induced by ω and $\tilde{\omega}$. Sections of \mathcal{V} bijectively correspond to P -equivariant functions $f : \mathcal{G} \rightarrow \mathbb{V}$, while sections of $\tilde{\mathcal{V}}$ correspond to Q -equivariant functions $f : \mathcal{G} \rightarrow \mathbb{V}$. In particular, since $Q \subset P$, every section of \mathcal{V} gives rise to a section of $\tilde{\mathcal{V}}$, and we can view $\Gamma(\mathcal{V}) \subset \Gamma(\tilde{\mathcal{V}})$.

Conversely, the proof of Proposition 3.3 in [ČG08] applied to our setting shows that a section $\tilde{s} \in \Gamma(\tilde{\mathcal{V}})$ is contained in $\Gamma(\mathcal{V})$ (i.e. the corresponding Q -equivariant function is actually P -equivariant) iff $\nabla_{\xi}^{\tilde{\mathcal{V}}} \tilde{s} = 0$ for all ξ in the vertical bundle of $\tilde{M} \rightarrow M$. The proof further shows that the tractor connection $\nabla^{\tilde{\mathcal{V}}}$ restricts to a connection on $\Gamma(\mathcal{V}) \subset \Gamma(\tilde{\mathcal{V}})$, which coincides with $\nabla^{\mathcal{V}}$. This implies a bijective correspondence between $\nabla^{\tilde{\mathcal{V}}}$ -parallel tractors in $\Gamma(\tilde{\mathcal{V}})$ and $\nabla^{\mathcal{V}}$ -parallel tractors in $\Gamma(\mathcal{V})$. If \mathbb{V} is irreducible as a \tilde{G} -representation but has a G -invariant subspace $\mathbb{W} \subset \mathbb{V}$, then this correspondence restricts to a bijective correspondence between parallel sections of $\tilde{\mathcal{W}} = \mathcal{G} \times_Q \mathbb{W} \rightarrow \tilde{M}$ and parallel sections of $\mathcal{W} = \mathcal{G} \times_P \mathbb{W} \rightarrow M$.

4.4. Exceptional case: Dimension two. In the special case of a projective structure in dimension $n = 2$ the curvature function of a normal projective Cartan connection takes values in $\Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{p}_+$, see e.g. [ČS09]. It is easily seen from the explicit matrices that $\mathfrak{p}_+ \subset \tilde{\mathfrak{p}} \cap \mathfrak{g}$. We can thus apply Proposition 3.1 in this case, which shows:

Proposition 4.2. *Suppose we are given a normal parabolic geometry (\mathcal{G}, ω) encoding a two-dimensional projective structure. Then the associated conformal parabolic geometry $(\tilde{\mathcal{G}}, \tilde{\omega})$ is normal, and thus $\nabla^{\tilde{\mathcal{V}}, \text{nor}} = \nabla^{\tilde{\mathcal{V}}}$ for any tractor bundle $\tilde{\mathcal{V}}$.*

This has some immediate consequences (compare with the results in [NS03], [DT10], [ČG10]):

Proposition 4.3. *The split-signature conformal structures obtained from two-dimensional projective structures via the Fefferman-type construction 4.1 have the following properties:*

- (1) *The conformal holonomy $\text{Hol}(\nabla^{\tilde{T}, \text{nor}})$ is contained in $\text{SL}(3)$.*
- (2) *The normal conformal tractor connection $\nabla^{\tilde{T}, \text{nor}}$ preserves the decomposition $\tilde{T} = \tilde{\mathcal{E}} \oplus \tilde{\mathcal{F}}$.*
- (3) *The adjoint tractor \mathbf{K} is parallel with respect to the normal tractor connection, i.e. $\nabla^{\Lambda^2 \tilde{T}, \text{nor}} \mathbf{K} = 0$. Thus \mathbf{K} corresponds to a normal conformal Killing field $k \in \mathfrak{X}(\tilde{M})$, i.e. an infinitesimal conformal isometry that inserts trivially into Weyl-curvature and Cotton-tensor (cf. [ČG08]).*
- (4) *The spin tractor bundle has two sections \mathbf{s}_E and \mathbf{s}_F with non-trivial pairing that are parallel with respect to the normal tractor connection, i.e. $\nabla^{\tilde{S}_+, \text{nor}} \mathbf{s}_E = 0$ and $\nabla^{\tilde{S}_-, \text{nor}} \mathbf{s}_F = 0$. Thus they correspond to two pure twistor spinors $\chi_e \in \Gamma(S_+[\frac{1}{2}])$ and $\chi_f \in \Gamma(S_-[\frac{1}{2}])$.*

4.4.1. *Almost Einstein structures.* A nice application of the construction in dimension two is that it makes visible the properties of a projective structure that correspond to the existence of almost Einstein scales of the associated conformal structure.

Proposition 4.4. *Suppose $(\tilde{M}, [g])$ is a conformal structure of signature $(2, 2)$ associated to a 2-dimensional projective structure $(M, [D])$ via the Fefferman-type construction.*

- (1) *Then $\mathbf{aEs}(\mathcal{C}) = \mathbf{aRs}([D]) \oplus \ker \Theta_0^T$.*
- (2) *Let $g \in \mathbf{aEs}(\mathcal{C})$ be defined on the open-dense subset $U \subset M$ and let D^g be the Levi-Civita connection of g . Then g corresponds to an almost Ricci-flat structure of $[D]$ if and only if $D^g \chi_f = 0$ and g corresponds to an element of $\ker \Theta_0^T$ if and only if $D^g \chi_e = 0$. In both cases it automatically follows that $\text{Ric}(g) = 0$.*

Proof. (1) We apply the relations between projective and conformal parallel tractors discussed above in section 4.3 to the conformal standard tractor bundle $\tilde{T} = \tilde{\mathcal{G}} \times_{\tilde{P}} \mathbb{R}^{3,3}$. As a $G = \text{SL}(3)$ representation $\mathbb{R}^{3,3}$ decomposes as $\mathbb{R}^{3,3} = \mathbb{R}^3 \oplus \mathbb{R}^{3*}$, and thus the conformal standard tractor bundle \tilde{T} decomposes. For a $\nabla^{\tilde{T}} = \nabla^{\tilde{\mathcal{E}}} + \nabla^{\tilde{\mathcal{F}}}$ parallel section $\tau = \tau_E + \tau_F$ of $\tilde{T} = \tilde{\mathcal{E}} \oplus \tilde{\mathcal{F}}$, one summand corresponds to a parallel section of the projective standard tractor bundle $\mathcal{E} = \mathcal{T} = \mathcal{G} \times_P \mathbb{R}^3$, and the other summand corresponds to a parallel section of the dual bundle $\mathcal{F} = \mathcal{T}^* = \mathcal{G} \times_P \mathbb{R}^{3*}$ (both equipped with the normal tractor connections $\nabla^{\mathcal{T}}$ and $\nabla^{\mathcal{T}^*}$).

Now by Proposition 4.2 we have $\nabla^{\tilde{T}} = \nabla^{\tilde{T}, \text{nor}}$. It is well known that parallel conformal standard tractors for the normal tractor connection correspond to almost Einstein structures $\mathbf{aEs}(\mathcal{C})$, see (11). $\nabla^{\mathcal{T}^*}$ -parallel projective co-tractors correspond to almost Ricci-flat structures $\mathbf{aRs}([D])$, see (7), and $\nabla^{\mathcal{T}}$ -parallel projective standard tractors correspond to solutions of the projectively invariant differential operator Θ_0^T , see (5).

- (2) A parallel conformal standard tractor $s \in \Gamma(\tilde{T})$ corresponds to an almost Ricci-flat scale of $(M, [D])$ iff lies in $\Gamma(\tilde{\mathcal{F}})$. On the other hand, parallel

standard tractors $\Gamma(\tilde{\mathcal{T}})$ correspond to almost Einstein scales, so we have to characterise those $\sigma \in \mathbf{aEs}(\mathcal{C})$ with $L_0^{\tilde{\mathcal{T}}} \sigma \in \Gamma(\tilde{\mathcal{F}})$. Since $\tilde{\mathcal{F}}$ was defined as the kernel of $\mathbf{s}_B \in \Gamma(\tilde{\mathcal{S}})$ under Clifford multiplication, we equivalently have to check when

$$L_0^{\tilde{\mathcal{T}}}(\sigma) \cdot \mathbf{s}_F = 0. \quad (29)$$

Now $U = M \setminus \sigma^{-1}(\{0\})$ is open-dense in M , hence suffices by continuity to verify (29) on that subset. On U we can use the Einstein metric g corresponding to the scale σ . Then according to (8)

$$s = L_0^{\tilde{\mathcal{T}}}(\sigma) = \begin{pmatrix} -\frac{1}{4}J \\ 0 \\ 1 \end{pmatrix},$$

where $J = g^{pq} P_{pq}$ is the trace of the Schouten tensor. Using

$$\mathbf{s}_F = L_0^{\tilde{\mathcal{S}}}(\chi_f) = \begin{pmatrix} \frac{1}{2\sqrt{2}}\not{D}\chi_f \\ \chi_f \end{pmatrix}$$

and the formula (12) for the tractor-Clifford action, equation (29) becomes

$$\begin{pmatrix} -\frac{1}{2\sqrt{2}}J\chi_f \\ -\frac{1}{2}\not{D}\chi_f \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since χ_f satisfies the twistor equation $D_c\chi_f + \frac{1}{4}\not{D}\chi_f = 0$, vanishing of $\not{D}\chi_f$ implies that χ_f is parallel with respect to D . In that case, since \mathbf{s}_F is $\nabla^{\tilde{\mathcal{S}}}$ -parallel,

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \nabla_c^{\tilde{\mathcal{S}}} \mathbf{s}_F = \nabla_c^{\tilde{\mathcal{S}}} L_0^{\tilde{\mathcal{S}}} \chi_f = \begin{pmatrix} \frac{1}{\sqrt{2}} P_{cp} \gamma^p \\ 0 \end{pmatrix},$$

and thus, since g is Einstein, $J = 0$ and $\text{Ric}(g) = 0$. This shows that (29) holds for $\sigma \in \mathbf{aEs}(\mathcal{C})$ if and only if $D\chi_f = 0$ on U , and then $\text{Ric}(g) = 0$ follows automatically.

The discussion for the case where g corresponds to an element in $\ker \Theta_0^{\mathcal{T}}$ is completely analogous. □

Remark 4.1. If $\mathbf{aRs}([D]) \neq \{0\}$ then $[D]$ is locally projectively flat, and therefore also \mathcal{C} is locally conformally flat: For a projective 2-dimensional structure the Weyl curvature as defined in (3) always vanishes and the Cotton-tensor $A(D)$ as defined in (4) is projectively invariant and the complete obstruction against projective flatness. If $\hat{D} \in \mathbf{aRs}([D])$ is a Ricci-flat affine connection on a open-dense subset $U \subset M$ then $\text{Ric}(\hat{D}) = 0$ implies that the Cotton-tensor $A(\hat{D})$ of \hat{D} vanishes on U . If $D \in [D]$, then by projective invariance $A(D) = A(\hat{D}) = 0$ on U , and by continuity thus $A(D) = 0$ on all of M .

4.4.2. Conformal Killing fields. Note that under $\mathfrak{sl}(3)$ the Lie algebra $\mathfrak{so}(3,3)$ decomposes into the following irreducible pieces

$$\mathbb{R}^3 \oplus \mathbb{R}^{3*} \oplus \mathfrak{sl}(3) \oplus \mathbb{R}. \quad (30)$$

Analogously to [ČG08, HS09] one can prove that:

Proposition 4.5. *The space of conformal Killing fields decomposes as*

$$\mathbf{aEs}(\mathcal{C}) \oplus \mathbf{inf}([D]) \oplus \mathbb{R}k, \quad (31)$$

where k is the conformal Killing field from Proposition 4.3, $\mathbf{aEs}(\mathcal{C})$ a subspace isomorphic to the space of almost Einstein structures, i.e. solutions of (10), and $\mathbf{inf}([D])$ a subspace isomorphic to the space of infinitesimal automorphisms of the original projective structure.

4.5. Remark: The construction for Lagrange contact structures.

Note that we can add an intermediate step to the construction of section 4.1. Let P' be the parabolic in G that stabilises the ray \mathbb{R}_+v_+ and the n -dimensional subspace \bar{E} , i.e. matrices of the form

$$\begin{pmatrix} a & Z^t & b \\ 0 & A & Y \\ 0 & 0 & c \end{pmatrix}. \quad (32)$$

Then obviously $Q \subset P' \subset P$. The correspondence space $M' = \mathcal{G} \times_P P/P'$ can be identified with the projectivised cotangent bundle $\mathcal{P}(T^*M)$. The parabolic geometry $(\mathcal{G} \rightarrow M', \omega)$ of type (G, P') defines a Lagrange contact structure on $\mathcal{P}(T^*M)$, i.e. a contact distribution $\mathcal{H} \subset TM'$ and a decomposition $\mathcal{H} = e' \oplus f'$ into two rank n subbundles such that the restriction of the Levi bracket to $e' \times e'$ and $f' \times f'$ vanishes identically (see e.g. [ČS09]). Hence the construction of section 4.1 can be regarded as the composition of a correspondence space construction from projective to Lagrange contact structures with a Fefferman-type construction from Lagrange contact to conformal structures, which is similar to the original Fefferman construction; one deals with different real forms of the same complex Lie groups in the two cases.

Proposition 4.6. *The Fefferman-type construction for Lagrange contact structures produces a normal conformal parabolic geometry iff the parabolic geometry encoding the Lagrange contact structure is torsion-free.*

Proof. If the geometry is torsion-free, then there is only one non-trivial harmonic curvature component (cf. [ČS09]) and ∂_1^* and ∂_2^* vanish separately on κ_H , and thus on κ . The harmonic curvature component κ_H takes values in $\Lambda^2(\mathfrak{g}/\mathfrak{p}')^* \otimes (\mathfrak{g}'^{ss} \oplus \mathfrak{p}'_+)$ (see e.g. [ČŽ09]). This is a P' submodule, and so the entire curvature takes values in that subspace. Since $\mathfrak{g}'^{ss} \oplus \mathfrak{p}'_+ \subset \mathfrak{g} \cap \tilde{\mathfrak{p}}$ we can apply Proposition 3.1 to conclude normality. The converse direction is obvious since every normal conformal geometry is torsion-free and $\mathfrak{g} \cap \tilde{\mathfrak{p}} \subset \mathfrak{p}'$. \square

Thus, as in the two dimensional case discussed before, we have:

Corollary 4.7. *For the split-signature conformal structures coming from torsion-free Lagrange contact structures*

- (1) *the conformal holonomy is contained in $\mathrm{SL}(n+1)$,*
- (2) *the normal conformal tractor connection $\nabla^{\tilde{T}, \mathrm{nor}}$ preserves the decomposition $\tilde{T} = \tilde{\mathcal{E}} \oplus \tilde{\mathcal{F}}$,*
- (3) *the adjoint tractor \mathbf{K} is parallel with respect to the normal conformal tractor connection and thus it corresponds to a normal conformal Killing field,*

- (4) the spin tractor bundle has two parallel sections $\mathbf{s}_E \in \Gamma(\tilde{\mathcal{S}}_+)$ and $\mathbf{s}_F \in \Gamma(\tilde{\mathcal{S}}_-)$ with non-trivial pairing, and these correspond to two pure twistor spinors $\chi_e \in \Gamma(S_\pm[\frac{1}{2}])$, $\chi_f \in \Gamma(S_-[\frac{1}{2}])$.

4.6. The projective construction for higher dimensions. For $n > 2$ the curvature of a normal projective Cartan connection is still contained in $\Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{p}$ but not in $\Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{p}_+$, and we cannot invoke Proposition 3.1 to conclude that the induced conformal Cartan connection is normal.

Proposition 4.8. *For $n > 2$ The conformal Cartan connection form $\tilde{\omega} \in \Omega^1(\tilde{\mathcal{G}}, \tilde{\mathfrak{g}})$ induced by the normal projective Cartan connection form $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ is normal if and only if ω is flat, in which case also $\tilde{\omega}$ is flat.*

Proof. If the induced conformal geometry is normal, then it is torsion-free, i.e. the curvature function $\tilde{\kappa}$ takes values in $\Lambda^2(\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes (\tilde{\mathfrak{p}} \cap \tilde{\mathfrak{g}})$. But this is only possible if the harmonic curvature of the original projective geometry takes values in a P -submodule of $\Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{p}/\mathfrak{p}_+$ that is contained in $\Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes (\tilde{\mathfrak{p}} \cap \mathfrak{g})/\mathfrak{p}_+$, and there is no such non-trivial P -invariant subspace. \square

Remark 4.2. To relate this to the previous section, note: a Lagrange contact structure coming from a projective structure via a correspondence space construction is torsion-free iff it is flat, or equivalently, iff the projective structure is flat (see e.g. [ČS09]).

4.6.1. Kostant codifferential of the curvature. In the non-flat case we need to understand how the normalised Cartan connection form $\tilde{\omega}^{nor}$ differs from $\tilde{\omega}$. As a preliminary step for the normalisation to be carried out in the proof of Theorem 4.11, we investigate the special form of

$$\tilde{\partial}^* \tilde{\kappa} : \mathcal{G} \rightarrow (\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \tilde{\mathfrak{p}} \cong (\mathfrak{g}/\mathfrak{q})^* \otimes \tilde{\mathfrak{p}},$$

and

$$\partial^* \tilde{\kappa}_0 : \mathcal{G} \rightarrow (\mathfrak{g}/\mathfrak{q})^* \otimes \tilde{\mathfrak{p}}/\tilde{\mathfrak{p}}_+$$

(i.e. the composition of $\partial^* \tilde{\kappa}$ with the projection $\tilde{\mathfrak{p}} \rightarrow \tilde{\mathfrak{p}}/\tilde{\mathfrak{p}}_+ = \tilde{\mathfrak{g}}_0$).

Proposition 4.9. *Suppose $\tilde{\omega}$ is the conformal Cartan connection induced from a normal projective Cartan connection via the Fefferman-type construction. Then, for any $u \in \mathcal{G}$, $\tilde{\partial}^* \tilde{\kappa}(u)$ can be viewed as an element in*

$$f \otimes \Lambda^2 \bar{F} \tag{33}$$

and $\tilde{\partial}^* \tilde{\kappa}_0(u)$ defines an element in

$$f \otimes \Lambda^2 f. \tag{34}$$

In this proof, we will often use the identification of $(\mathbb{R}^{n+1, n+1})^*$ with $\mathbb{R}^{n+1, n+1}$ provided by the bilinear form (20), which identifies E^* with F .

Proof. A priori, $\tilde{\partial}^* \tilde{\kappa}(u)$ is an element of $(\mathfrak{g}/\mathfrak{q})^* \otimes \tilde{\mathfrak{p}}$. Since elements of \mathfrak{p} insert trivially into $\tilde{\kappa}(u)$ we have that $\partial^* \tilde{\kappa}(u)$ annihilates $f = \mathfrak{p}/\mathfrak{q}$, and since f is maximally isotropic $\partial^* \tilde{\kappa}(u)$ can thus be viewed as an element in $f \otimes \tilde{\mathfrak{p}}$.

Next we determine the subspace of $\tilde{\mathfrak{p}} \subset \Lambda^2 \mathbb{R}^{n+1, n+1}$ where $\tilde{\partial}^* \tilde{\kappa}(u)(X)$ takes its values using (17). The space spanned by the elements \tilde{Z}_i , $i = 1, \dots, n$, can be characterised as the annihilator of the vertical space $f = \mathfrak{p}/\mathfrak{q}$, i.e. it

is the space of all $\tilde{Z} \in \tilde{\mathfrak{p}}_+$ such that $B(\tilde{Z}, X) = 0$ for all $X \in \mathfrak{p}$, where B denotes the Killing form. One can easily see from the explicit form of \mathfrak{p} and \tilde{v}_+ (see (24) and (22)) that the image of the action of \mathfrak{p} on \tilde{v}_+ is $\bar{F} + L$. Furthermore, the action of an element $Z \in \tilde{\mathfrak{p}}_+$ annihilates \tilde{v}_+ and maps $X \cdot \tilde{v}_+ \in \bar{F} + L$ to (a multiple) of $B(Z, X)\tilde{v}_+$. Thus the subspace spanned by the $\tilde{Z}_i, i = 1, \dots, n$, is contained the annihilator of $\bar{F} + L$ in $\tilde{\mathfrak{g}}$. Note that $\bar{F} + L$ is a \mathfrak{p} submodule and so is the annihilator of that subspace. Since $\tilde{\kappa}(u)(X, X_i) \subset \mathfrak{p}$ this implies that $\tilde{\partial}^* \tilde{\kappa}(u)(X)$ annihilates $\bar{F} + L$.

Now, the \mathfrak{g} -module decomposition of $\tilde{\mathfrak{g}}$ looks as follows

$$\tilde{\mathfrak{g}} = \Lambda^2(E \oplus F) = \underbrace{(E \otimes F)_0}_{\mathfrak{g}} \oplus \underbrace{(E \otimes F)_{Tr} \oplus \Lambda^2 E \oplus \Lambda^2 F}_{\mathfrak{g}^\perp}$$

and in block-matrices

$$= \begin{pmatrix} E \otimes F & \Lambda^2 E \\ \Lambda^2 F & E \otimes F \end{pmatrix}.$$

The assumption that the projective Cartan connection be normal implies that $\tilde{\partial}^* \tilde{\kappa}(u)(X) \subset \mathfrak{g}^\perp$, by (17) and since $\tilde{Z}_i - Z_i \in \mathfrak{g}^\perp$. Vanishing of $\tilde{\partial}^* \tilde{\kappa}(u)(X)$ on \bar{F} and skew-symmetry implies that the $\Lambda^2 E$ -part of $\tilde{\partial}^* \tilde{\kappa}(u)(X)$ has to vanish. Vanishing on $\tilde{v}_+ = \pi_E(\tilde{v}_+) + \pi_F(\tilde{v}_+)$ and on $\pi_F(\tilde{v}_+) \subset \bar{F}$ implies vanishing on $\pi_E(\tilde{v}_+)$. But then $\tilde{\partial}^* \tilde{\kappa}(u)(X)$ has also trivial $(E \otimes F)_{Tr}$ -part and is indeed contained in the subspace of maps in $\Lambda^2 F$ that vanish on $\pi_E(\tilde{v}_+)$, i.e. in $\Lambda^2 \bar{F}$. Then $\tilde{\partial}^* \tilde{\kappa}(u)(X) \subset \Lambda^2 \bar{F}$ implies that $\tilde{\partial}^* \tilde{\kappa}_0(u)(X) \subset \Lambda^2 f$, which shows (34). \square

Remark 4.3. We have seen in the proof of Proposition 3.1 that $\tilde{\mathfrak{p}}_+ \cap \mathfrak{g}^\perp = \{0\}$, and thus the restriction of the projection $\tilde{\mathfrak{p}} \rightarrow \tilde{\mathfrak{p}}/\tilde{\mathfrak{p}}_+$ to the subspace $\tilde{\mathfrak{p}} \cap \mathfrak{g}^\perp$ is injective. Note that this implies that for every $\phi_0 \in f \otimes \Lambda^2 f$ there is a unique element $\phi \in f \otimes \Lambda^2 \bar{F} \subset f \otimes (\tilde{\mathfrak{p}} \cap \mathfrak{g}^\perp)$ that projects onto ϕ_0 .

4.6.2. Reduced Weyl-structures. As a technical preliminary to study how the normalised Cartan connection form $\tilde{\omega}^{nor}$ differs from $\tilde{\omega}$ we now relate the Weyl structures of the original Cartan geometry (\mathcal{G}, ω) and those of $(\tilde{\mathcal{G}}, \tilde{\omega})$:

Proposition 4.10. *Any projective Weyl structure*

$$\mathcal{G}_0 \xrightarrow{j} \mathcal{G}$$

induces a conformal Weyl structure

$$\mathcal{G}_0 \hookrightarrow \tilde{\mathcal{G}}_0 \xrightarrow{\tilde{j}} \tilde{\mathcal{G}}.$$

Proof.

$$Q_0 = G_0 \cap Q = G_0 \cap (G \cap \tilde{P}) = G_0 \cap \tilde{P} = G_0 \cap \tilde{G}_0.$$

We have $G_0 \cong P/P_+$, and since $P_+ \subset Q$, $Q_0 \cong Q/P_+$, and thus $G_0/Q_0 \cong G/P$. Therefore the reduction $\mathcal{G}_0 \xrightarrow{j} \mathcal{G}$ from P to G_0 over the manifold M induces a reduction from $Q \subset P \subset \tilde{P}$ to $Q_0 \subset G_0 \subset \tilde{Q}$ over \tilde{M} . Composing

the embedding $\mathcal{G}_0 \xrightarrow{j} \mathcal{G}$ with the natural embedding $\mathcal{G} \hookrightarrow \tilde{\mathcal{G}}$, one obtains a reduction

$$\mathcal{G}_0 \xrightarrow{\tilde{j}} \tilde{\mathcal{G}}$$

from \tilde{P} to $Q_0 \subset \tilde{G}_0$ over \tilde{M} . Let $\tilde{\mathcal{G}}_0 := \mathcal{G}_0 \times_{Q_0} \tilde{G}_0$, then the embedding $\mathcal{G}_0 \hookrightarrow \tilde{\mathcal{G}}_0$ is natural and \tilde{j} canonically extends to an embedding of the \tilde{G}_0 -bundle $\tilde{\mathcal{G}}_0$ into the \tilde{P} -bundle $\tilde{\mathcal{G}}$, which we simply denote by \tilde{j} again. We therefore see that \tilde{j} is a reduced Weyl structure of the conformal Cartan bundle $\tilde{\mathcal{G}}$. \square

A version of this result in a more general context has been proved in [Alt10].

4.6.3. Preserved spin-tractors and induced twistor spinors. Let $\mathbf{s}_F \in \Gamma(\tilde{\mathcal{S}}_-)$ be the spin tractor with kernel $\tilde{\mathcal{F}} \subset \tilde{\mathcal{T}}$ as in 4.2.

Theorem 4.11. $\mathbf{s}_F \in \Gamma(\mathcal{S}_-)$ is parallel with respect to the normal conformal spin tractor connection $\nabla^{\mathcal{S}_-, \text{nor}} \mathbf{s}_F = 0$. In particular, the conformal spin structure (M, \mathcal{C}) carries a canonical (pure) twistor spinor $\chi_f \in \Gamma(\mathcal{S}_-[\frac{1}{2}])$.

Proof. We are going to normalise the Cartan connection $\tilde{\omega} \in \Omega^1(\tilde{\mathcal{G}}, \tilde{\mathfrak{g}})$ that is induced by the projective Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$. Any other conformal Cartan connection $\tilde{\omega}'$ differs from $\tilde{\omega}$ by some $\Psi \in \Omega^1(\tilde{\mathcal{G}}, \tilde{\mathfrak{g}})$: $\tilde{\omega}' = \tilde{\omega} + \Psi$. This Ψ must vanish on vertical fields and be P -equivariant. The condition on $\tilde{\omega}'$ to induce the same conformal structure on M as $\tilde{\omega}$ is that Ψ has values in $\tilde{\mathfrak{p}} \subset \tilde{\mathfrak{g}}$. One can therefore regard Ψ as a P -equivariant function $\Psi : \tilde{\mathcal{G}} \rightarrow (\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \tilde{\mathfrak{p}}$.

The general theory of parabolic geometries, [ČS09], tells us that there is a unique such Ψ such that the curvature function $\tilde{\kappa}'$ of $\tilde{\omega}'$ satisfies $\tilde{\partial}^* \tilde{\kappa}' = 0$, and then $\tilde{\omega}'$ is the normal conformal Cartan connection $\tilde{\omega}^N$.

The normalisation of $\tilde{\omega}$ proceeds by homogeneity of $(\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \tilde{\mathfrak{p}}$, which decomposes into two homogeneous components according to the decomposition $\tilde{\mathfrak{p}} = \tilde{\mathfrak{g}}_0 \oplus \tilde{\mathfrak{p}}_+$. The failure of $\tilde{\omega}$ to be normal is $\tilde{\partial}^*(\tilde{\kappa}) : \mathcal{G} \rightarrow (\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \tilde{\mathfrak{p}}$. In the first step of normalisation one looks for a Ψ^0 such that $\tilde{\omega}^1 = \tilde{\omega} + \Psi^0$ has $\tilde{\partial}^* \tilde{\kappa}'$ taking values in the highest homogeneity $\tilde{\partial}^* \tilde{\kappa}' : \tilde{\mathcal{G}} \rightarrow (\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \tilde{\mathfrak{p}}_+$.

To write down this first normalisation it is useful to employ a Weyl structure $\tilde{\mathcal{G}}_0 \xrightarrow{j} \tilde{\mathcal{G}}$, and by Proposition 4.10 we can take a Weyl structure that is induced by a Q_0 -reduction

$$\mathcal{G}_0 \xrightarrow{j} \mathcal{G} \hookrightarrow \tilde{\mathcal{G}}.$$

This allows us to project $\tilde{\partial}^* \tilde{\kappa}$ to $(\tilde{\partial}^* \tilde{\kappa})_0 : \mathcal{G}_0 \rightarrow (\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \tilde{\mathfrak{g}}_0$ and to employ the \tilde{G}_0 -equivariant Kostant Laplacian $\tilde{\square} : (\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \tilde{\mathfrak{g}}_0 \rightarrow (\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \tilde{\mathfrak{g}}_0$. For the first normalisation step we need to form a map $\Psi^0 : \tilde{\mathcal{G}} \rightarrow (\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \tilde{\mathfrak{p}}$ that agrees with $-\tilde{\square}^{-1}(\tilde{\partial}^* \tilde{\kappa})_0$ in the $\tilde{\mathfrak{g}}_0$ -component. If we have formed any such Ψ^0 along $\mathcal{G}_0 \xrightarrow{j} \tilde{\mathcal{G}}$ we can just equivariantly extend this to all of $\tilde{\mathcal{G}}$.

Now $\tilde{\square}$ restricts to an invertible endomorphism of $((\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \tilde{\mathfrak{g}}_0) \cap \text{im } \tilde{\partial}^*$ that acts by scalar multiplication on each of the three \tilde{G}_0 -irreducible components of $((\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \tilde{\mathfrak{g}}_0) \cap \text{im } \tilde{\partial}^*$. To write down this decomposition we use that under

\tilde{G}_0 one has $\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}} \cong \mathbb{R}^{n,n}$. As a \tilde{G}_0 -module, $(\mathbb{R}^{n,n*} \otimes \tilde{\mathfrak{g}}_0) \cap \text{im } \tilde{\partial}^*$ decomposes into

$$((\mathbb{R}^{n,n*} \otimes \tilde{\mathfrak{g}}_0) \cap \text{im } \tilde{\partial}^*)_{tr} \oplus ((\mathbb{R}^{n,n*} \otimes \tilde{\mathfrak{g}}_0) \cap \text{im } \tilde{\partial}^*)_{alt} \oplus ((\mathbb{R}^{n,n*} \otimes \tilde{\mathfrak{g}}_0) \cap \text{im } \tilde{\partial}^*)_{\odot},$$

where

$$\begin{aligned} ((\mathbb{R}^{n,n*} \otimes \tilde{\mathfrak{g}}_0) \cap \text{im } \tilde{\partial}^*)_{tr} &= \mathbb{R}^{n,n}, \\ ((\mathbb{R}^{n,n*} \otimes \tilde{\mathfrak{g}}_0) \cap \text{im } \tilde{\partial}^*)_{alt} &= \Lambda^3 \mathbb{R}^{n,n} \end{aligned}$$

and $((\mathbb{R}^{n,n*} \otimes \tilde{\mathfrak{g}}_0) \cap \text{im } \tilde{\partial}^*)_{\odot}$ the highest weight component, which is the trace- and alternation-free part.

Now $\mathbb{R}^{n,n}$ has the Q_0 -invariant subspace f , and, it was shown in Proposition 4.9 that

$$\tilde{\partial}^* \tilde{\kappa}_0 \in f \otimes \Lambda^2 f.$$

This shows that $\tilde{\partial}^* \tilde{\kappa}_0$ has no trace-component, and since $\Lambda^3 f \subset f \otimes \Lambda^2 f$, we have that

$$\begin{aligned} (\tilde{\partial}^* \tilde{\kappa}_{\tilde{\mathfrak{g}}_0})_{tr} &= 0, \\ (\tilde{\partial}^* \tilde{\kappa}_{\tilde{\mathfrak{g}}_0})_{alt} &\in f \otimes \Lambda^2 f \text{ and} \\ (\tilde{\partial}^* \tilde{\kappa}_{\tilde{\mathfrak{g}}_0})_{\odot} &\in f \otimes \Lambda^2 f. \end{aligned}$$

Since $\tilde{\square}$ preserves these components it follows that also

$$-\tilde{\square}^{-1} \tilde{\partial}^* \tilde{\kappa}_0 \in f \otimes \Lambda^2 f.$$

For each element in $f \otimes \Lambda^2 f \subset (\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \tilde{\mathfrak{g}}_0$ there exists a unique element in $(\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \Lambda^2 \tilde{F} \subset (\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \tilde{\mathfrak{p}}$ with that $\tilde{\mathfrak{g}}_0$ -component, cf. Remark 4.3. This defines a canonical

$$\Psi^0 : \mathcal{G}_0 \rightarrow (\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \Lambda^2 \tilde{F} \subset (\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \tilde{\mathfrak{p}}$$

for the first normalisation step and we set $\tilde{\omega}^1 = \tilde{\omega} + \Psi^0$.

Since $F \subset \mathbb{R}^{n+1,n+1}$ is the kernel of the pure spinor $s_F \in \Delta_-^{n+1,n+1}$ we see that the tractor spinor s_F induced by the constant map

$$\mathcal{G} \rightarrow \Delta_-^{n+1,n+1}, \quad u \mapsto s_F$$

is still parallel with respect to $\tilde{\omega}^1$.

Now we want to see that also after the second normalisation step, which yields the normal conformal Cartan connection $\tilde{\omega}^2 = \tilde{\omega}^{nor}$, the tractor spinor s_F is still parallel. One has $\tilde{\omega}^{nor} = \tilde{\omega}^1 + \Psi^1$, with $\Psi^1 : \mathcal{G}_0 \rightarrow \tilde{\mathfrak{p}}_+$, and we denote the spin tractor connections on $\tilde{\mathcal{S}}_-$ induced by $\tilde{\omega}^1$ and $\tilde{\omega}^{nor}$ by $\nabla^{\tilde{\mathcal{S}}_-,1}$ resp. $\nabla^{\tilde{\mathcal{S}}_-,nor}$.

Recall from 2.4.2 that $[\tilde{\mathcal{S}}_-]_g = \begin{pmatrix} S_+[-\frac{1}{2}] \\ S_-[\frac{1}{2}] \end{pmatrix}$. Now $\nabla^{\tilde{\mathcal{S}}_-,1} s_F = 0$ and

$$(\nabla^{\tilde{\mathcal{S}}_-,nor} s_F - \nabla^{\tilde{\mathcal{S}}_-,1} s_F) = \Psi^1 s_F \in \Gamma(S_+[-\frac{1}{2}]) \subset \Gamma(\tilde{\mathcal{S}}_-),$$

and therefore

$$\nabla^{\tilde{\mathcal{S}}_-,nor} s_F \in \Gamma(S_+[-\frac{1}{2}]). \quad (35)$$

Let $\mathbf{s}_F = \begin{pmatrix} \tau \\ \chi_f \end{pmatrix} \in \Gamma(\tilde{\mathcal{S}}_-)$. Then (35) says explicitly that

$$\begin{pmatrix} D_c \tau + \frac{1}{\sqrt{2}} P_{cp} \gamma^p \chi_f \\ D_c \chi_f + \frac{1}{\sqrt{2}} \gamma_c \tau \end{pmatrix} = \begin{pmatrix} * \\ 0 \end{pmatrix}.$$

It follows in particular that χ_f is a twistor spinor, and necessarily $\tau = \frac{1}{\not{D}} \chi_f$ - which shows that $\mathbf{s}_F = L_0^{\tilde{\mathcal{S}}_-}(\chi_f)$. But then $D_c \tau + \frac{1}{\sqrt{2}} P_{cp} \gamma^p \chi_f = 0$ is a differential consequence of that equation, and thus indeed

$$0 = \nabla^{\tilde{\mathcal{S}}_-, nor}(L_0^{\tilde{\mathcal{S}}_-} \chi_f) = \nabla^{\tilde{\mathcal{S}}_-, nor} \mathbf{s}_F.$$

□

Since $\text{Hol}(\mathcal{C}) = \text{Hol}(\nabla^{\tilde{\mathcal{S}}_-, nor})$ Proposition 4.11 in particular implies that the induced conformal structures have reduced holonomy:

Corollary 4.12. *The conformal holonomy $\text{Hol}(\mathcal{C})$ is contained in the isotropy subgroup of $\mathbf{s}_F \in \Delta_-^{n+1, n+1}$ in $\text{Spin}(n+1, n+1)$; this is $SL(n+1) \ltimes \Lambda^2(\mathbb{R}^{n+1})^* \subset \text{Spin}(n+1, n+1)$.*

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